

C12 - 9.0 - Rationals Review

Holes: Factor the top, Factor the bottom. If a factor cancels, there is a hole when the factor equals zero.

$$y = \frac{\cancel{x-3}}{(x-3)(x+2)} = \frac{1}{x+2} \quad x-3=0 \quad x=3 \quad y = \frac{1}{(3)+2} = \frac{1}{5} \quad \text{Hole: } \left(3, \frac{1}{5}\right) \quad \text{Domain Restriction: } x \neq 3$$

Vertical Asymptote: denominator = 0

$R: y \neq \text{hole}$

$$y = \frac{1}{x+1} \quad x+1=0 \quad x=-1 \quad \text{Set denominator equal to zero and solve.}$$

VA: $x = -1$

Domain Restriction: $x \neq \text{VA or Holes}$

VA

x	y
-2	-1
-1	und
0	1

Horizontal Asymptote:

Point on Both sides of VA
(And between VA's)

Case 1:

Case 2:

Case 3:

$$\begin{array}{lll} x^2, \frac{x^2}{x} & \frac{1}{x}, \frac{1}{x^2} & HA: y = 0 \\ HA: \text{none} & \frac{1}{x} + c, \frac{1}{x^2} + c & HA: y = c \end{array} \quad \begin{array}{ll} \frac{3x^2}{2x^2} & HA: y = \frac{3}{2} \\ \frac{3x^2}{2x^2} + c & HA: y = \frac{3}{2} + c \end{array}$$

x	y
$-\infty$?
∞	?

Range Restrictions: $y \neq \text{HA or Holes}$

Intercepts
x - intercepts: Set $y = 0$ and Solve $(x, 0)$
y - interceptes: Set $x = 0$ and Solve $(0, y)$

Close to asymptote
Through point/s
Close to asymptote

Slant Asymptote: Do Synthetic or Long Division and if the Quotient, the Answer, is a linear function that is the equation of the slant asymptote. (Case #1)

Graph: Holes, VA, HA, TOV, *x* - int, *y* - int, SA

$\frac{2}{x-1} + 3$	$\frac{2}{x-1} + 3 = \frac{2}{x-1} + 3 \times \frac{x-1}{x-1} = \frac{2}{x-1} + \frac{3x-3}{x-1} = \frac{3x-1}{x-1}$	$\begin{array}{r} 3 \\ x-1 \overline{) 3x-1} \\ \underline{3x-3} \\ 2 \end{array}$	$\begin{array}{r} 3x-1 \\ x-1 = 0 \\ x = +1 \quad +1 \overline{) 3-1} \\ \underline{+3} \\ 3 \end{array}$	$\frac{2}{x-1} + 3 = \frac{3x-1}{x-1}$
	$3 + \frac{2}{x-1}$	$3 \text{ R: } 2$		

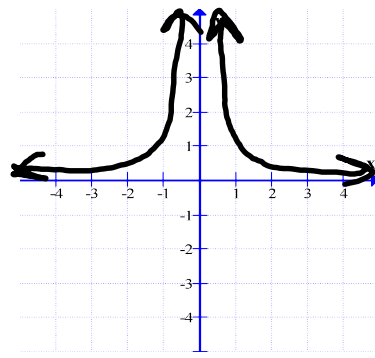
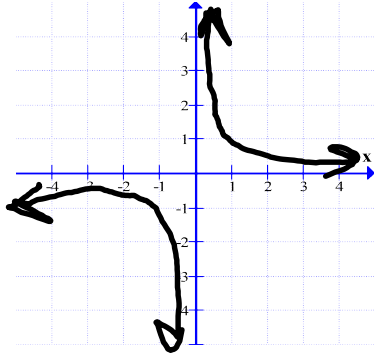
$$y = \frac{a}{x-h} + k$$

$$y = \frac{a}{VA} + HA$$

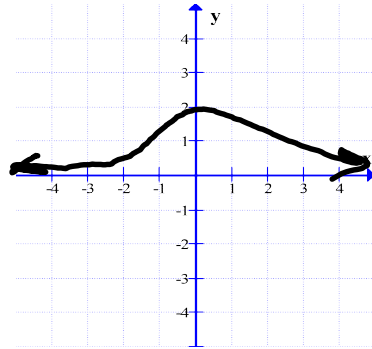
$$y = \frac{a(HA)(x-int)(holes)}{(HA)(VA's)(holes)}$$

C12 - 9.0 - Rationals Graphs

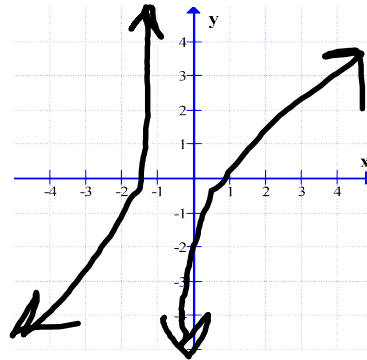
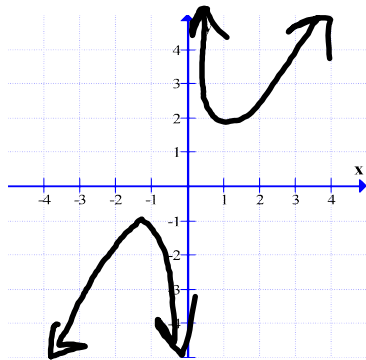
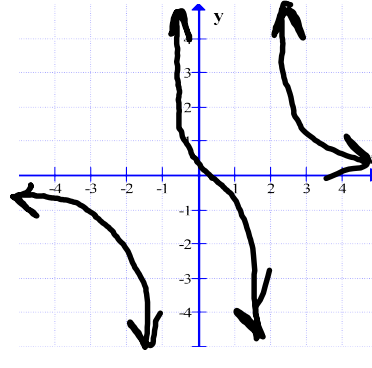
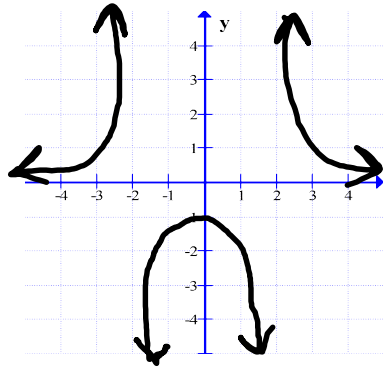
$$y = \frac{1}{x}$$



$$y = \frac{1}{x^2}$$

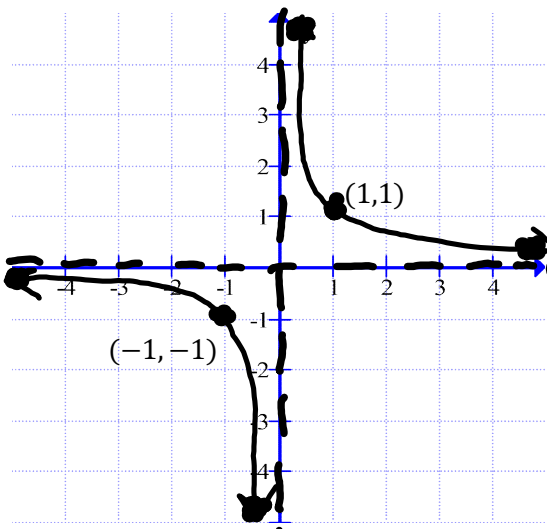


$$y = \frac{1}{x^2 + 1}$$



$$y = \frac{a}{x-h} + k$$

C12 - 9.1 - Graph TOV HT xy-int Notes



x	y
-5	$-\frac{1}{5} = -0.2$
-1	-1
$-\frac{1}{10} = -0.1$	-10
0	und
$\frac{1}{10} = 0.1$	10
1	1
5	$\frac{1}{5}$

$$y = \frac{1}{x}$$

$$y = \frac{1}{(-\frac{1}{10})}$$

$$y = 1(-\frac{10}{1})$$

$$y = -10$$

$$y = \frac{1}{x}$$

$$HA: y = 0$$

$$HA: y = k$$

$$VA: x = 0$$

VA: Set Denominator = 0 and solve

$$x = 0 \quad \text{Domain: } x \neq 0$$

x-int:

$$y = \frac{1}{x}$$

$$0 = \frac{1}{x}$$

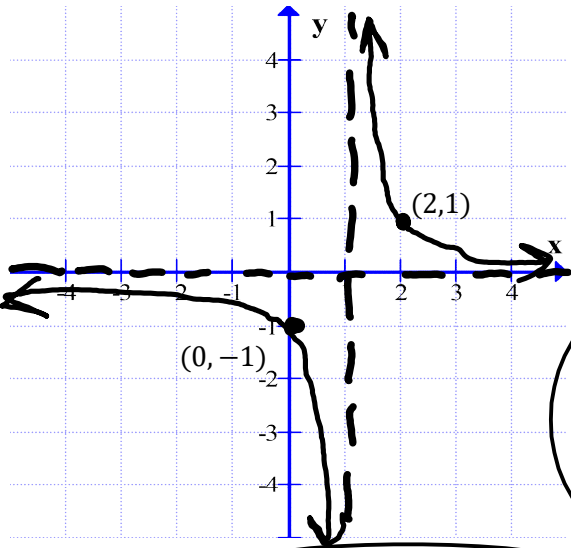
$$0 \neq 1$$

y-int:

$$y = \frac{1}{x}$$

$$y = \frac{1}{0}$$

$$y \neq 0$$



$$HT = +1$$

$$y = \frac{1}{x-1}$$

$$HA: y = 0$$

Range

$$y \neq 0$$

$$VA: x = 1 \quad \text{Domain: } x \neq 1$$

End Behavior

$$x \rightarrow \infty, \quad y \rightarrow 0^+$$

$$x \rightarrow -\infty, \quad y \rightarrow 0^-$$

As x gets close to ...

Behavior near Asymptote

$$x \rightarrow 1^+, \quad y \rightarrow \infty$$

$$x \rightarrow 1^-, \quad y \rightarrow -\infty$$

VA:

$$x - 1 = 0$$

$$x = 1$$

x-int:

$$y = \frac{1}{x-1}$$

$$0 = \frac{1}{x-1}$$

$$0 = \frac{1}{x-1} \times (x-1)$$

Careful!

$$(x-1) \times 0 = \frac{1}{x-1} \times (x-1)$$

$$0 \neq 1$$

y-int:

$$y = \frac{1}{x-1}$$

$$y = \frac{1}{0-1}$$

$$y = -1$$

$$(0, -1)$$

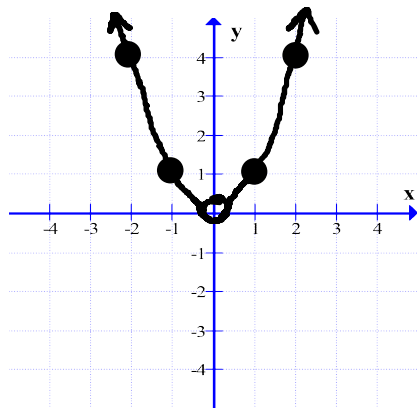
x	y
0	-1
1	und
2	1

C12 - 9.1 - Horizontal Asymptotes Cases Notes

$y = \frac{ax^m}{bx^n}$	$m > n$	HA: none	
	$m < n$	HA: $y = 0$	or HA: $y = c$
	$m = n$	HA: $y = \frac{a}{b}$	or HA: $y = \frac{a}{b} + c$

Case 1:

$$y = \frac{x^3}{x^1}$$

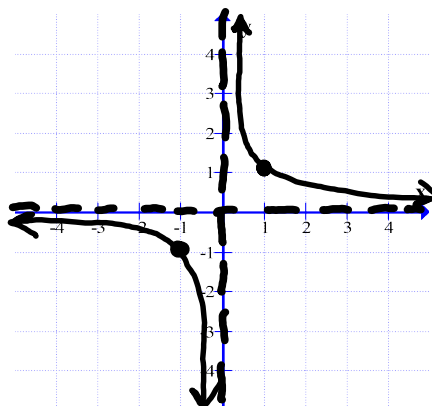


If the exponent of x is higher on the top than the bottom

HA: none

Case 2:

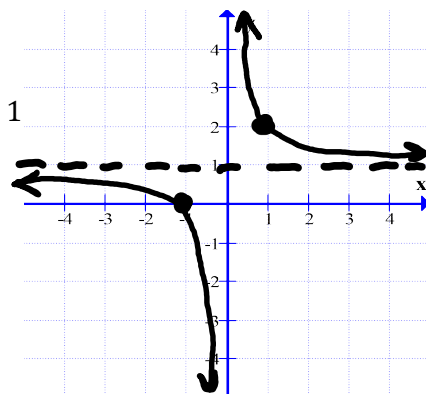
$$y = \frac{x^1}{x^2}$$



If the exponent of x is higher on the bottom

HA: $y = 0$

$$y = \frac{x^1}{x^2} + 1$$



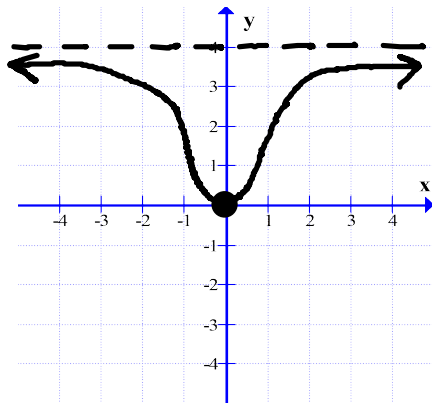
If case 2 is shifted up or down = c

HA: $y = c$ $y = 1$

$\frac{x^1}{x^2} + 1 = \frac{1x^1 + 1x^2}{1x^2}$ <p>LCD</p>

Case 3:

$$y = \frac{4x^2}{1x^2 + 2}$$

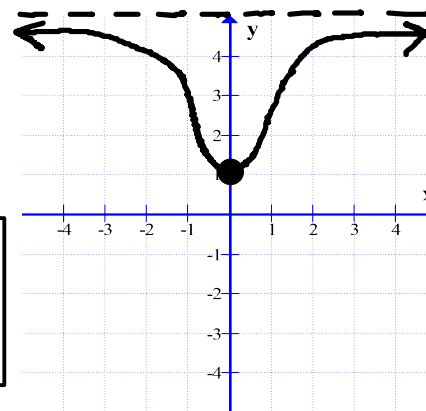


If the exponent of x is the same on the top as the bottom

HA: $y = \text{fraction of coefficients}$

$$HA: y = \frac{4}{1}$$

$$y = \frac{4x^2}{1x^2 + 1} + 1$$



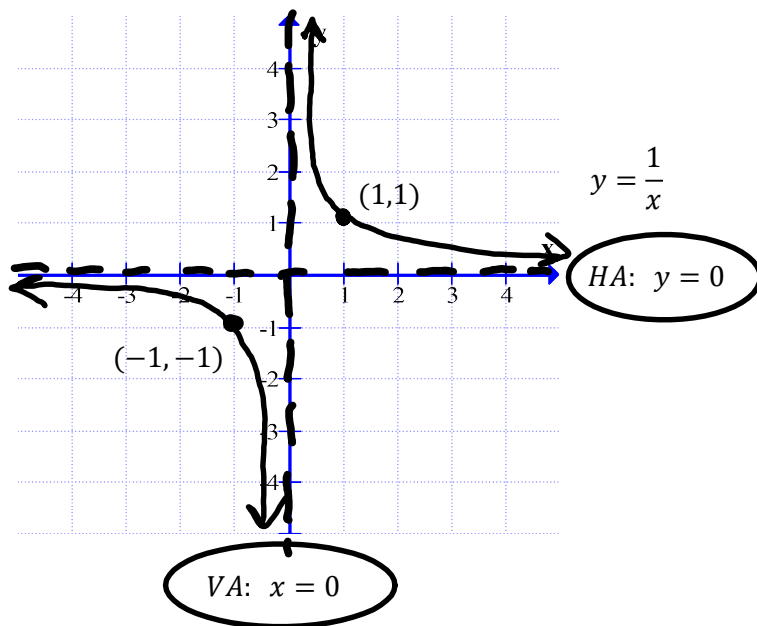
If case 3 is shifted up or down = c

HA: $y = \text{fraction of coefficients} + c$

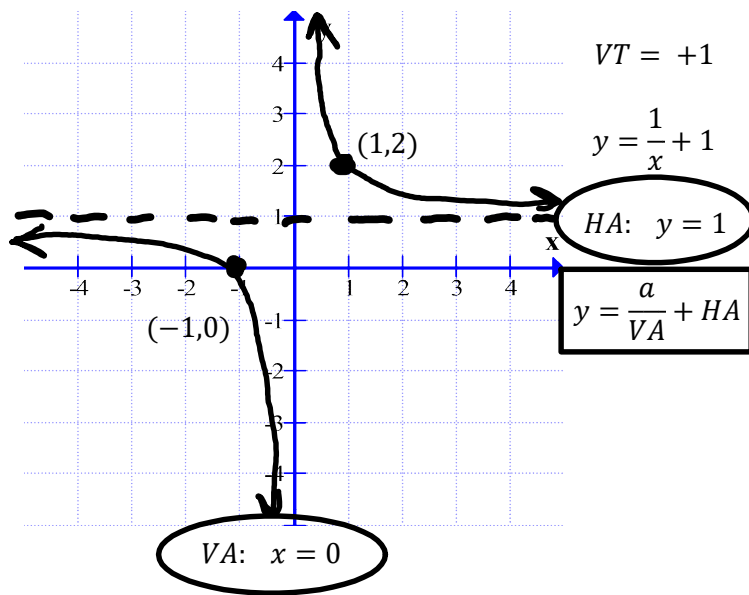
$$HA: y = \frac{4}{1} + 1 = 5$$

$\frac{4x^2}{1x^2 + 1} + 1 = \frac{5x^2 + 1}{1x^2 + 1}$ <p>LCD</p>
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C12 - 9.2 - Graph VT Add Fractions Long Division Notes



x	y
-1	-1
0	und
1	1



Add Fractions

$$\frac{1}{x} + 1$$

LCD

$$\frac{1}{x} + 1 \times \frac{x}{x}$$

$$\frac{1}{x} + \frac{x}{x}$$

$$\frac{1+x}{x}$$

$y = \frac{x+1}{x}$

Long Division

$$x \overline{) x+1}$$

1
↑
remainder

$$\frac{x+1}{x} = 1 + \frac{1}{x}$$

$\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$

$$\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$$

Separate Fractions

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$x - \text{int:}$

$$y = \frac{1}{x} + 1$$

$$0 = \frac{1}{x} + 1$$

$$-1 = \frac{1}{x}$$

$$-1x = 1$$

$$x = -1$$

$(-1, 0)$

$$y = \frac{x+1}{x}$$

$$0 = \frac{x+1}{x}$$

$$x \times 0 = \frac{x+1}{x} \times x$$

$$0 = x+1$$

$$x = -1$$

$y - \text{int:}$

$$y = \frac{1}{x} + 1$$

$$y = \frac{1}{0} + 1$$

$y \neq$

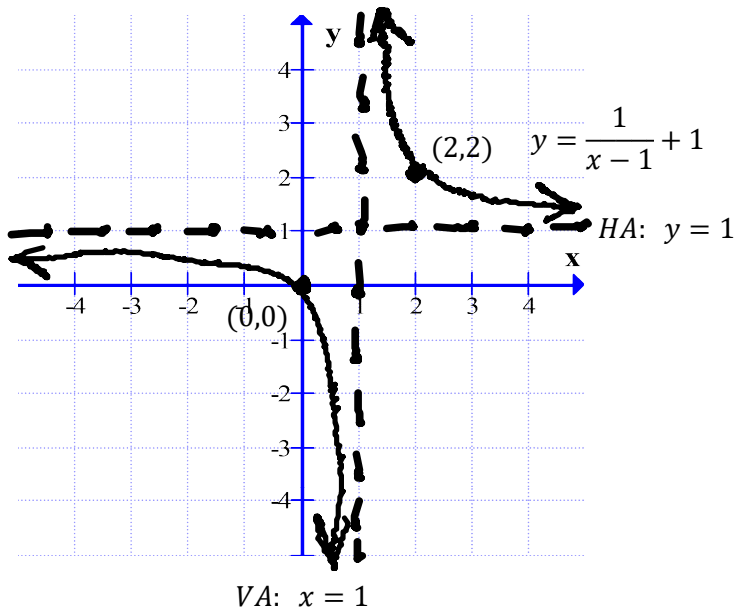
$$y = \frac{x+1}{x}$$

$$y = \frac{0+1}{0}$$

$y \neq$

x	y
-1	0
0	und
1	2

C12 - 9.2 - Graph HT VT Add Fractions Long Div Notes



$$\frac{1}{x-1} + 1$$

$$\frac{1}{x-1} + 1 \times \frac{x-1}{x-1}$$

$$\frac{1}{x-1} + \frac{x-1}{x-1}$$

$$\frac{1+x-1}{1+x-1}$$

$$\frac{(x-1)(x-1)}{x}$$

$$\frac{x}{x-1}$$

$$y = \frac{x}{x-1}$$

$$x-1 \overline{) \frac{x+0}{x-1}}$$

$$\frac{x}{x-1} = \frac{1}{x-1} + 1$$

$x - int:$

$$y = \frac{1}{x-1} + 1$$

$$0 = \frac{1}{x-1} + 1$$

$$-1 = \frac{1}{x-1}$$

$$(x-1) \times -1 = \frac{1}{x-1} \times (x-1)$$

$$-x + 1 = 1$$

$$x = 0$$

Careful!

$$y = \frac{x}{x-1}$$

$$0 = \frac{x}{x-1}$$

$$(x-1) \times 0 = \frac{x}{x-1} \times (x-1)$$

$$0 = x$$

$$x = 0$$

$(0,0)$

$y - int:$

$$y = \frac{1}{x-1} + 1$$

$$y = \frac{x}{x-1}$$

$$y = \frac{1}{0-1} + 1$$

$$y = \frac{0}{0-1}$$

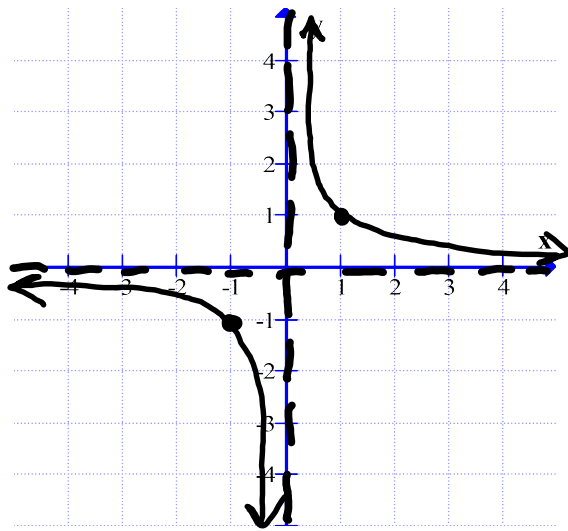
$$y = -1 + 1$$

$$y = 0$$

$(0,0)$

x	y
0	0
1	und
2	2

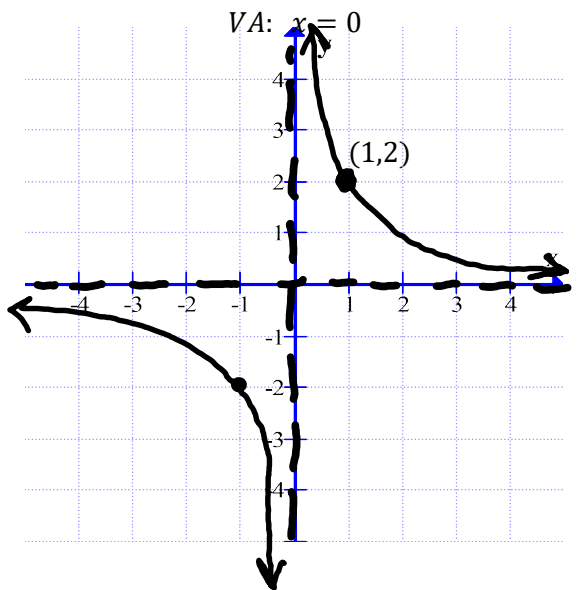
C12 - 9.3 - Graph VE VR Notes



$$y = \frac{1}{x}$$

$$HA: y = 0$$

x	y
-1	1
0	<i>und</i>
1	1



$$VA: x = 0$$

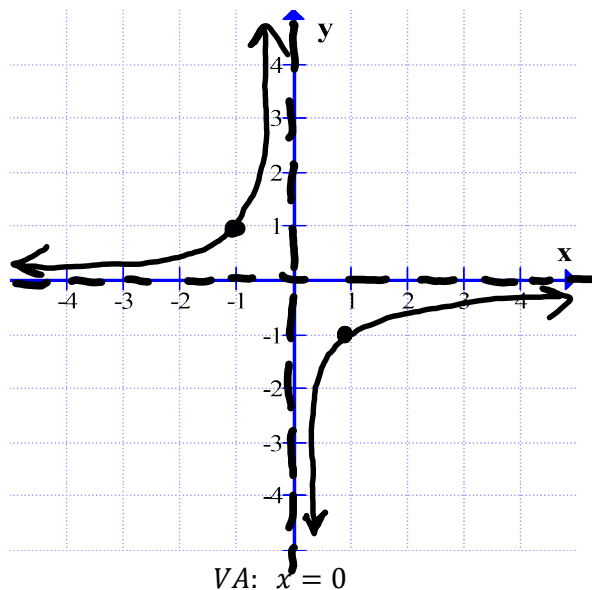
$$VE = 2$$

$$y = \frac{2}{x}$$

$$HA: y = 0$$

x	y
-1	-2
0	<i>und</i>
1	2

$$VA: x = 0$$



VR

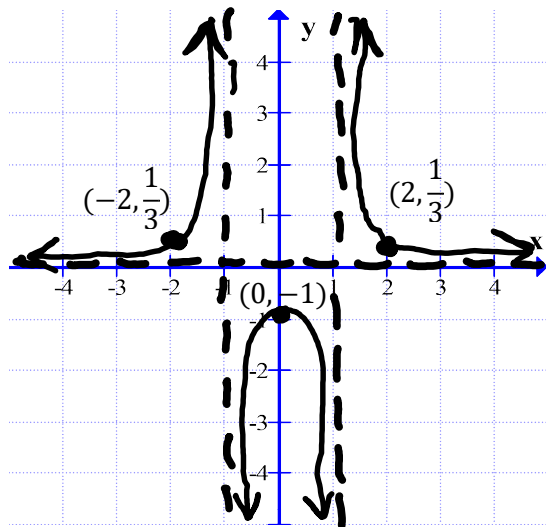
$$y = -\frac{1}{x}$$

$$HA: y = 0$$

x	y
-1	1
0	<i>und</i>
1	-1

$$VA: x = 0$$

C12 - 9.4 - Graph 2xVA's Notes



$$y = \frac{1}{x^2 - 1}$$

$$HA: y = 0$$

$$VA: x = -1 \quad VA: x = 1$$

$$VA: \quad x^2 - 1 = 0 \\ (x + 1)(x - 1) = 0$$

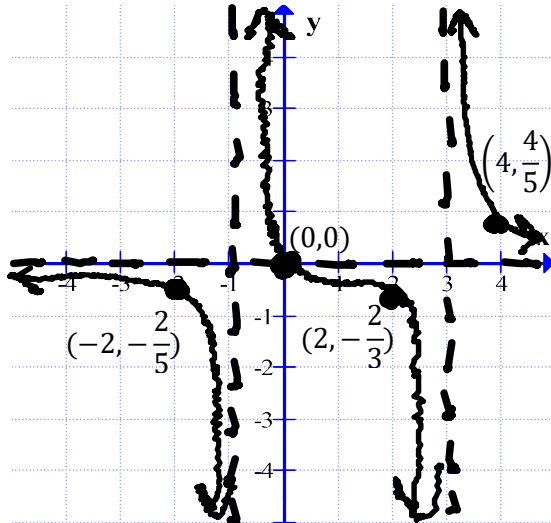
$$x + 1 = 0 \quad x - 1 = 0 \\ x = -1 \quad x = 1$$

$$x - int: \quad y = \frac{1}{x^2 - 1} \\ 0 = \frac{1}{x^2 - 1} \\ 0 \neq 1$$

$$y - int: \quad y = \frac{1}{x^2 - 1} \\ y = \frac{1}{0^2 - 1} \\ y = -1$$

$$(0, -1)$$

x	y
-2	$\frac{1}{3}$
-1	und
0	-1
1	und
2	$\frac{1}{3}$



$$y = \frac{x}{x^2 - 2x - 3}$$

$$HA: y = 0$$

$$VA: x = -1 \quad VA: x = 3$$

$$VA: \quad x^2 - 2x - 3 = 0 \\ (x + 1)(x - 3) = 0$$

$$x + 1 = 0 \quad x - 3 = 0 \\ x = -1 \quad x = 3$$

$$x - int:$$

$$0 = \frac{x}{x^2 - 2x - 3} \\ 0 = x \\ x = 0$$

$$y - int:$$

$$y = \frac{0}{0^2 - 2(0) - 3} \\ y = 0$$

$$(0, 0)$$

x	y
-2	$-\frac{2}{5}$
-1	und
0	0
2	$-\frac{2}{3}$
3	und
4	$\frac{4}{5}$

C12 - 9.5 - Holes Notes

$$y = \frac{(x-1)(x+2)}{x+2}$$

~~$$y = \frac{(x-1)(x+2)}{x+2}$$~~

$$y = x - 1$$

Hole: $x + 2 = 0$
 $x = -2$

Set what you've crossed off equal to zero and solve.

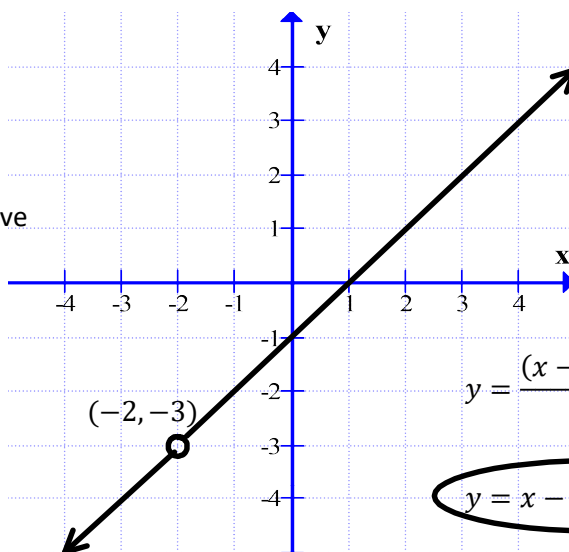
$$y = x - 1$$

$$y = -2 - 1$$

$$y = -3$$

$$(-2, -3)$$

x	y
-2	-3



$$y = \frac{(x-1)(x+2)}{x+2}$$

$$y = x - 1 \quad x \neq -2$$

$$x \neq -2$$

$$y = \frac{x+3}{(x-1)(x+3)}$$

~~$$y = \frac{x+3}{(x-1)(x+3)}$$~~

$$y = \frac{1}{x-1}$$

Hole: $x + 3 = 0$
 $x = -3$

VA: $x - 1 = 0$
 $x = 1$

$$y = \frac{x+3}{(x-1)(x+3)}$$

$$y = \frac{1}{x-1} \quad x \neq -3$$

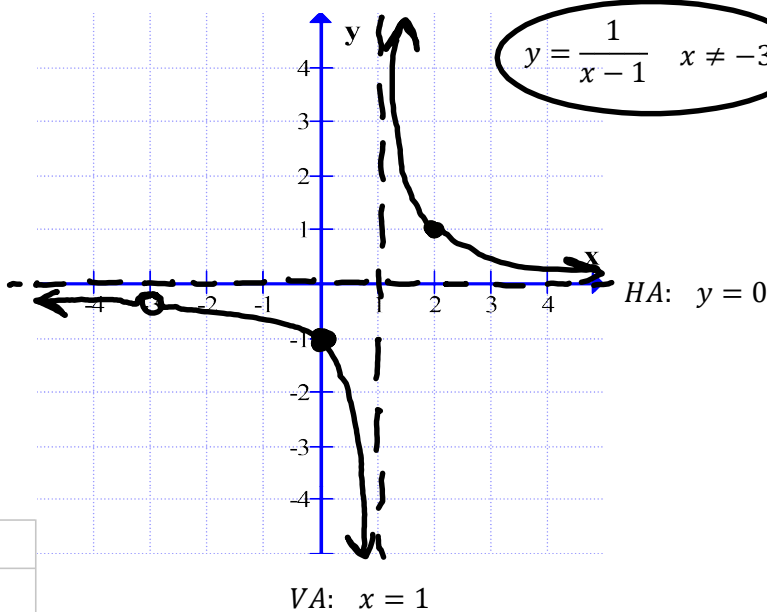
$$y = \frac{1}{x-1}$$

$$y = \frac{1}{(-3)-1}$$

$$y = \frac{1}{-4}$$

$$\left(-3, -\frac{1}{4}\right)$$

x	y
-3	$-\frac{1}{4}$



VA: $x = 1$

C12 - 9.6 - Slant Asymptote Notes

$$y = \frac{x^2}{x+1}$$

VA: $x + 1 = 0$
 $x = -1$ HA: $\frac{x^2}{x}$ none

Slant Asymptote

$$\begin{array}{r} x-1 \\ x+1 \overline{) x^2 + 0x + 0} \\ \underline{-(x^2 + x)} \\ -x + 0 \\ \underline{-(-x - 1)} \\ +1 \end{array}$$

$$\begin{array}{r} x^2 \\ x+1 = 0 \\ x = -1 \\ + \begin{array}{r} 1 \\ \downarrow -1 1 \\ \hline 1 1 1 \end{array} \end{array}$$

Slant Asymptote

$$y = x - 1$$

$$x - 1 \quad R: +1$$

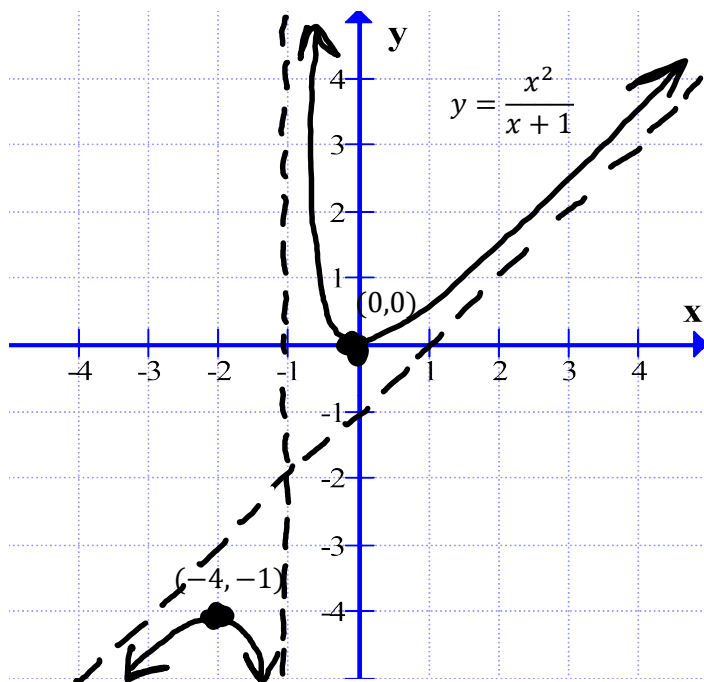
$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

$$x - 1 + \frac{1}{x+1}$$

$$x - 1 + \frac{1}{x+1}$$

$$\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$$

$$\text{Slant} + \frac{R}{\text{Divisor}}$$



Slant Asymptote

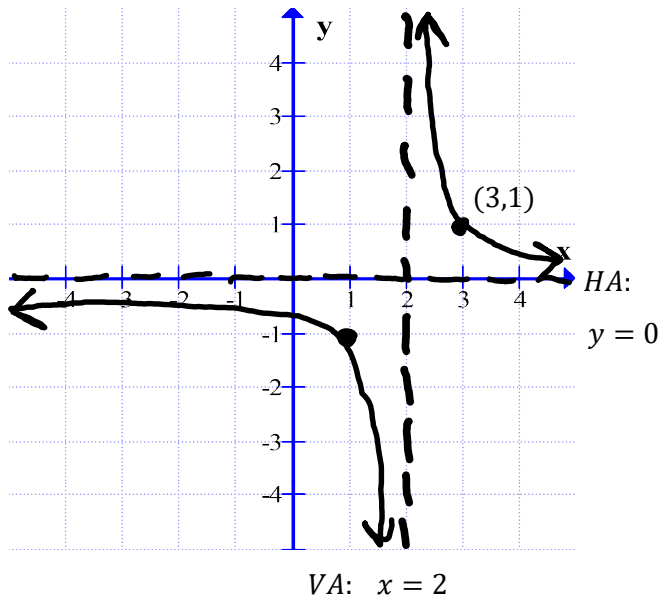
$$y = x - 1$$

x	y
-4	-1
-1	und
0	0

VA: $x = -1$

$$\begin{aligned} \frac{x^2}{x+1} &= x - 1 + \frac{1}{x+1} \\ x - 1 &\times \frac{x+1}{x+1} + \frac{1}{x+1} \\ &\frac{x^2 - 1 + 1}{x+1} \\ &\frac{x^2}{x+1} \end{aligned}$$

C12 - 9.7 - HT/VT Graph -> Equation Notes



$$y = \frac{a}{x - h} + k$$

$$y = \frac{a}{x - 2} + k$$

$$y = \frac{a}{x - 2} + 0$$

$$y = \frac{a}{x - 2}$$

$$1 = \frac{a}{3 - 2}$$

$$a = 1$$

$$y = \frac{1}{x - 2}$$

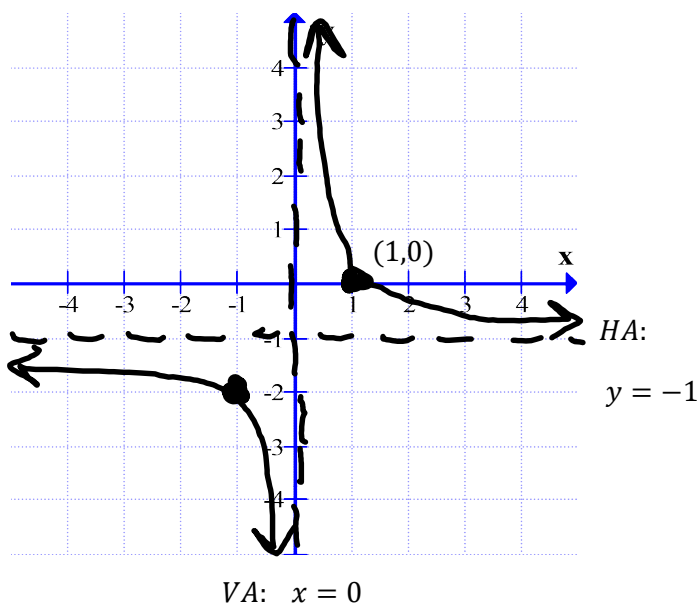
$$y = \frac{a}{VA's} + HA$$

$$x = 2 \quad x - 2 = 0 \quad VA: x = 2$$

$$k = 0 \quad HA: y = 0$$

$$HA: y = k$$

$(3, 1)$
 (x, y)



$$y = \frac{a}{x - h} + k$$

$$y = \frac{a}{x - 0} + k$$

$$y = \frac{a}{x} + k$$

$$y = \frac{a}{x} - 1$$

$$0 = \frac{a}{1} - 1$$

$$a = 1$$

$$y = \frac{1}{x} - 1$$

$$y = \frac{a}{VA's} + HA$$

$$x = 0 \quad VA: x = 0$$

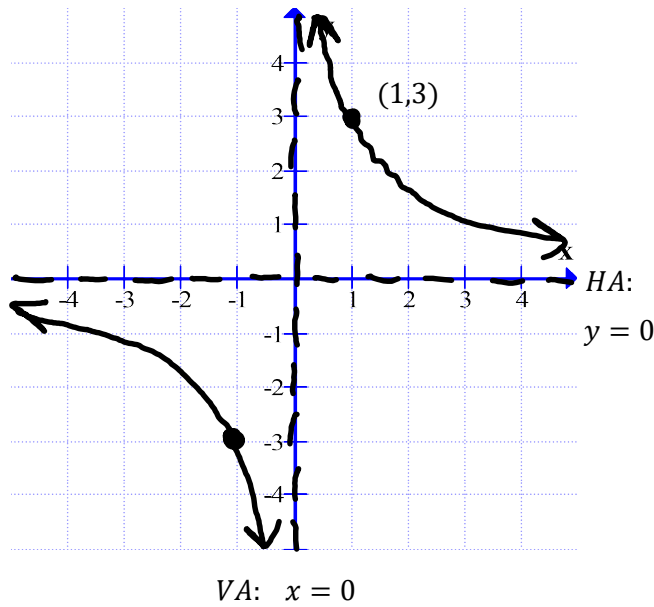
$$k = -1 \quad HA: y - 1$$

$$HA: y = k$$

$(1, 0)$
 (x, y)

$y = \frac{a(x - r)}{x - h}$	$y = \frac{HA(x - int)}{VA's}$		
$y = \frac{a(x - r)}{x}$	$VA: x = 0$		
$y = \frac{a(x - 1)}{x}$	$x = 1$	$x: int: (1, 0)$	
	$x - 1 = 0$		
$y = \frac{-(x - 1)}{x}$	$HA: y = -1$	Case 3: $\frac{-1x}{1x}$	
			$\frac{1}{x} - 1$
			$\frac{1}{x} - 1 \times \frac{x}{x}$
			$\frac{1}{x} - \frac{x}{x}$
			$\frac{1 - x}{x}$
			$y = -\frac{x - 1}{x}$
			Add Fractions: LCD

C12 - 9.7 - E/C/R Graph -> Equation Notes



$$y = \frac{a}{x-h} + k$$

$$y = \frac{a}{x} + k \quad x = 0 \quad VA: x = 0$$

$$y = \frac{a}{x} + 0 \quad k = 0 \quad HA: y = 0$$

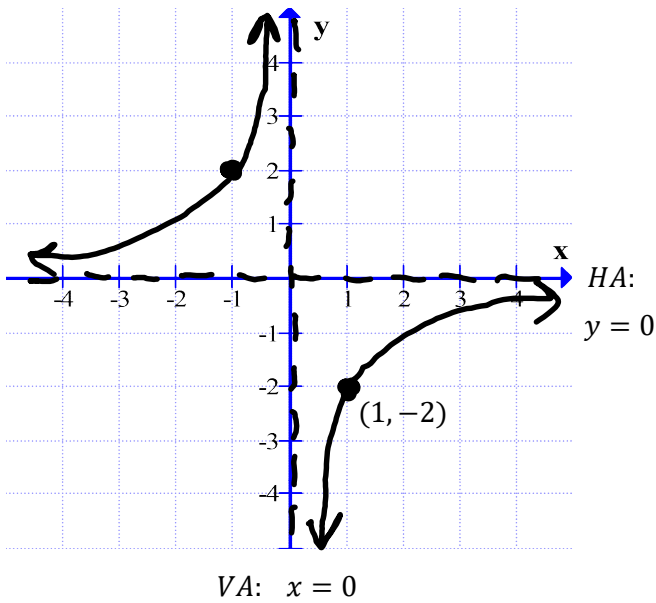
$$y = \frac{a}{x} \quad HA: y = k$$

$$3 = \frac{a}{1} \quad (1, 3)$$

$$(x, y)$$

$$a = 3$$

$$y = \frac{3}{x}$$



$$y = \frac{a}{x-h} + k$$

$$y = \frac{a}{x-0} + k \quad x = 0 \quad VA: x = 0$$

$$y = \frac{a}{x} + k$$

$$y = \frac{a}{x} + 0 \quad k = 0 \quad HA: y = 0$$

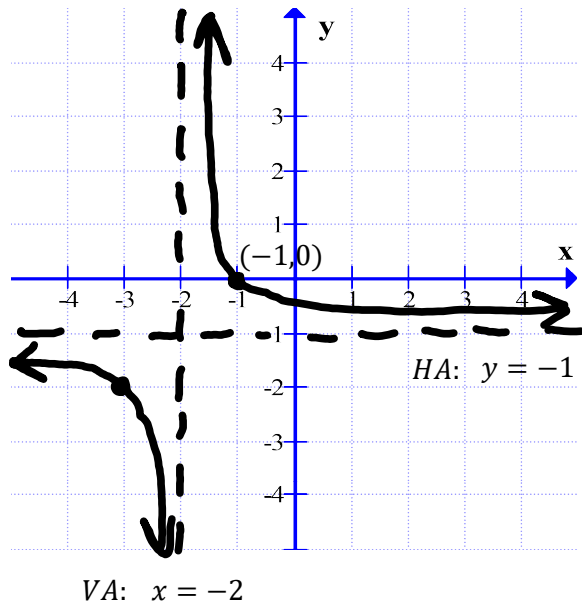
$$-2 = \frac{a}{1} \quad (1, -2)$$

$$(x, y)$$

$$a = -2$$

$$y = \frac{-2}{x}$$

C12 - 9.7 - HT/VT Combo Graph -> Equation Notes



$$y = \frac{a}{x-h} + k$$

$$y = \frac{a}{x+2} + k \quad x = -2 \quad VA: x = -2$$

$$x + 2 = 0$$

$$y = \frac{a}{x+2} - 1 \quad k = -1 \quad HA: y = -1$$

$$HA: y = k$$

$$y = \frac{a}{x+2} - 1$$

$$0 = \frac{a}{-1+2} - 1 \quad (-1, 0)$$

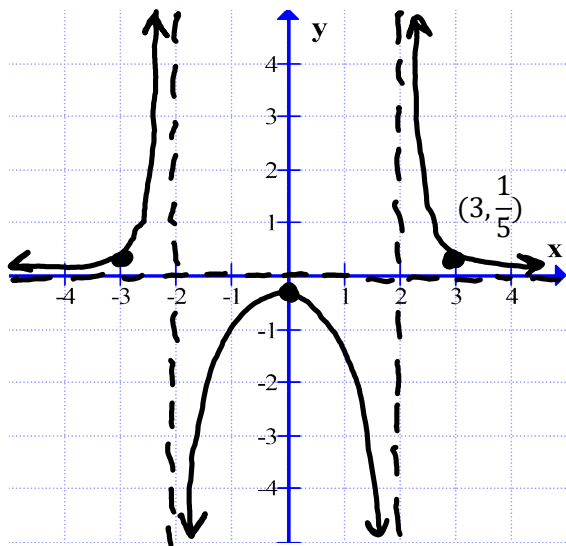
$$(x, y)$$

$$a = 1$$

$$y = \frac{1}{x+2} - 1$$

$y = \frac{a(x-r)}{x-h}$	$y = \frac{HA(x-int)}{VA's}$	
$y = \frac{a(x-r)}{x+2}$	$VA: x = -2$	$\frac{1}{x+2} - 1$
$y = \frac{a(x+1)}{x+2}$	$x = -1 \quad x - int: (-1, 0)$	$\frac{1}{x+2} - 1 \times \frac{x+2}{x+2}$
	$x + 1 = 0$	$\frac{1}{x+2} - \frac{x+2}{x+2}$
$y = \frac{-(x+1)}{x+2}$	$HA: y = -1$	$\frac{-x-1}{x+2}$
	$Case 3: \frac{-1x}{1x}$	$y = \frac{-(x+1)}{x+2}$

C12 - 9.7 - HT/VT Combo Graph -> Equation Notes



VA: $x = -2$ VA: $x = 2$

HA:
 $y = 0$

$$y = \frac{a}{x - h} + k$$

$$y = \frac{a}{(x + 2)(x - 2)} + k$$

$$y = \frac{a}{(x + 2)(x - 2)} + 0$$

$$y = \frac{a}{(x + 2)(x - 2)}$$

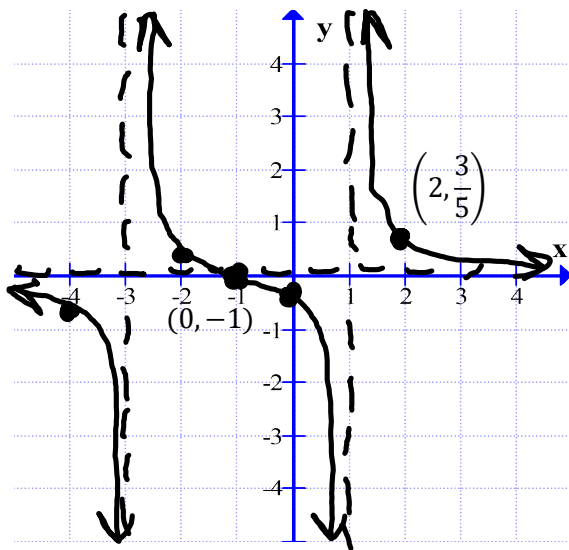
$$\frac{1}{5} = \frac{a}{(3 + 2)(3 - 2)} \quad \left(3, \frac{1}{5}\right)$$

(x, y)

$$a = 1$$

$$y = \frac{1}{(x + 2)(x - 2)}$$

$$y = \frac{1}{x^2 - 4}$$



VA: $x = -3$ VA: $x = 1$

HA:
 $y = 0$

$$y = \frac{a}{x - h} + k$$

$$y = \frac{a(x - int)}{VA's} + HA$$

$$y = \frac{a(x + 1)}{(x + 3)(x - 1)} + k$$

$$y = \frac{a(x + 1)}{(x + 3)(x - 1)} + 0$$

$$y = \frac{a(x + 1)}{(x + 3)(x - 1)}$$

$$\frac{3}{5} = \frac{a(2 + 1)}{(2 + 3)(2 - 1)} \quad \left(2, \frac{3}{5}\right)$$

(x, y)

$$\frac{3}{5} = \frac{3a}{5}$$

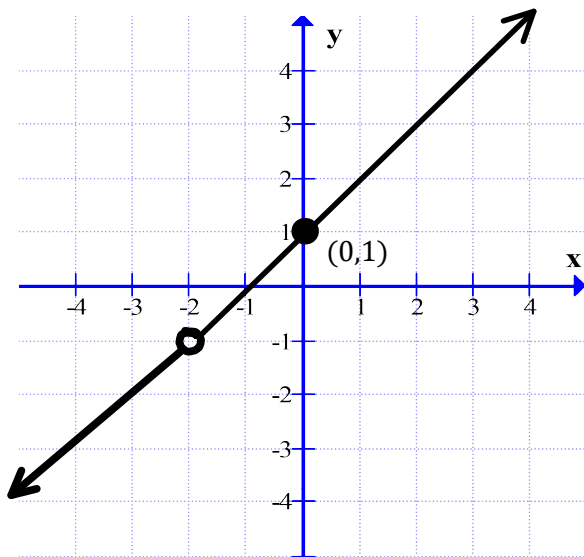
$$a = 1$$

$$y = \frac{x + 1}{(x + 3)(x - 1)}$$

$$y = \frac{a(x - r)}{x - h} \quad \boxed{y = \frac{HA(x - int)}{VA's}}$$

$$y = \frac{(x + 1)}{(x + 3)(x - 1)} \quad HA: y = 0 \quad \text{Case 2: } \frac{x}{x^2}$$

C12 - 9.7 - Holes Graph -> Equation Notes



$$y = \frac{a}{x - h}$$

$$y = \frac{a(x - \text{int})(\text{holes})}{(VA's)(\text{holes})}$$

$$y = \frac{a(x + 2)}{(x + 2)} + k$$

$$x = -2 \quad \text{hole: } (-2, -1)$$

$$y = \frac{a(x + 1)(x + 2)}{(x + 2)}$$

$$y = \frac{a(x + 1)(x + 2)}{(x + 2)}$$

$$x = -1$$

$$x + 1 = 0 \quad x - \text{int: } (0, -1)$$

$$y = a(x + 1) \quad (0, 1)$$

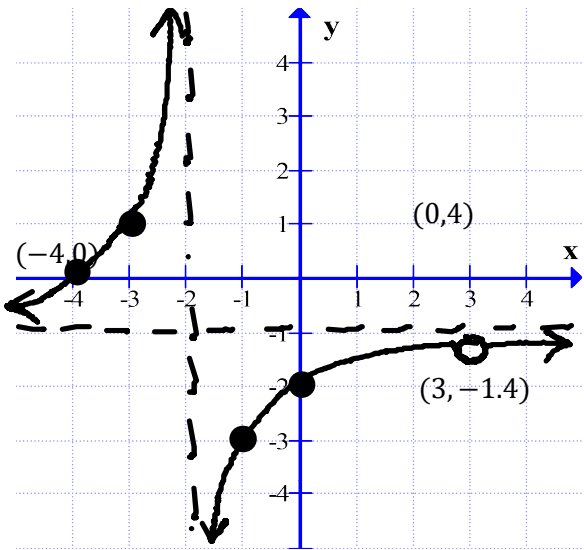
$$(x, y)$$

$$1 = a(0 + 1)$$

$$a = 1$$

$$y = \frac{(x + 2)(x + 1)}{(x + 2)}$$

HA: none



$$y = \frac{a}{x - h}$$

$$y = \frac{a(x - \text{int})(\text{holes})}{(VA's)(\text{holes})}$$

$$y = \frac{a(x - 3)}{(x - 3)}$$

$$x = 3 \quad \text{hole: } (3, -1.4)$$

$$x - 3 = 0$$

$$y = \frac{a(x - 3)}{(x + 2)(x - 3)}$$

$$x = -2 \quad VA: x = -2$$

$$x + 2 = 0$$

$$y = \frac{a(x + 4)(x - 3)}{(x + 2)(x - 3)}$$

$$x = -4 \quad x - \text{int: } (0, -4)$$

$$x + 4 = 0$$

$$y = \frac{a(x + 4)(x - 3)}{(x + 2)(x - 3)}$$

$$-2 = \frac{a(0 + 4)}{(0 + 2)} \quad (0, -2)$$

$$(x, y)$$

$$-2 = \frac{4a}{2}$$

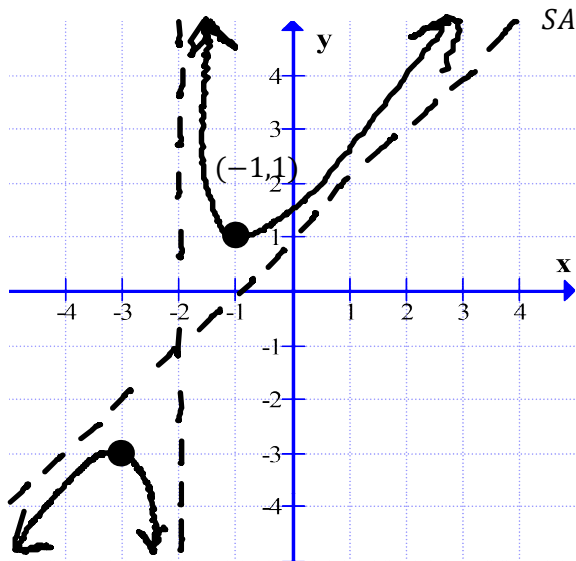
$$a = -1$$

$$y = \frac{-1(x + 4)(x - 3)}{(x + 2)(x - 3)}$$

HA: Case 3: $\frac{-1x^2}{1x^2}$

$$y = -\frac{1}{1}$$

C12 - 9.7 - Slant Graph -> Equation Notes



SA: $y = x + 1$

$$y = \frac{a}{x-h} + \text{Slant}$$

SA: $y = x + 1$

$$y = \frac{a}{x+2} + x + 1$$

$$y = \text{Slant} + \frac{R}{\text{Divisor}}$$

$$1 = \frac{a}{-1+2} - 1 + 1$$

(-1, 1)
(x, y)

$$a = a$$

$$y = \frac{1}{x+2} + x + 1$$

$$\frac{1}{x+2} + x + 1$$

$$\frac{1}{x+2} + x + 1 \times \frac{x+2}{x+2}$$

$$\frac{1}{x+2} + x + 1 \times \frac{x+2}{x+2}$$

$$\frac{1}{x+2} + \frac{x^2 + 3x + 2}{x+2}$$

$$\frac{x^2 + 3x + 3}{x+2}$$

$$y = \frac{x^2 + 3x + 3}{x+2}$$