2019

AP® CollegeBoard

AP® Calculus AB

Scoring Guidelines

Question 1

(a) $\int_0^5 E(t)\,dt = 153.457690$

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) The rate of change in the number of fish in the lake at time t is given by $E(t) - L(t)$.

$E(t) - L(t) = 0 \Rightarrow t = 6.20356$

$E(t) - L(t) > 0$ for $0 \le t < 6.20356$, and $E(t) - L(t) < 0$ for $6.20356 < t \le 8$. Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

— OR —

Let $A(t)$ be the change in the number of fish in the lake from midnight to t hours after midnight.

$A(t) = \int_0^t (E(s) - L(s))\,ds$

$A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356$

t	$A(t)$
0	0
C	135.01492
8	80.91998

Therefore the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203).

$3 : \begin{cases} 1 : \text{sets } E(t) - L(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $E'(5) - L'(5) = -10.7228 < 0$

Because $E'(5) - L'(5) < 0$, the rate of change in the number of fish is decreasing at time $t = 5$.

$2 : \begin{cases} 1 : \text{considers } E'(5) \text{ and } L'(5) \\ 1 : \text{answer with explanation} \end{cases}$

Question 2

(a) v_P is differentiable $\Rightarrow$ v_P is continuous on $0.3 \le t \le 2.8$.

$$\frac{v_P(2.8) - v_P(0.3)}{2.8 - 0.3} = \frac{55 - 55}{2.5} = 0$$

By the Mean Value Theorem, there is a value c, $0.3 < c < 2.8$, such that $v_P{}'(c) = 0$.

— OR —

v_P is differentiable $\Rightarrow$ v_P is continuous on $0.3 \le t \le 2.8$.

By the Extreme Value Theorem, v_p has a minimum on $[0.3, 2.8]$.
$v_P(0.3) = 55 > -29 = v_P(1.7)$ and $v_P(1.7) = -29 < 55 = v_P(2.8)$.
Thus v_P has a minimum on the interval $(0.3, 2.8)$.

Because v_P is differentiable, $v_P{}'(t)$ must equal 0 at this minimum.

$$2 : \begin{cases} 1 : v_P(2.8) - v_P(0.3) = 0 \\ 1 : \text{justification, using} \\ \quad \text{Mean Value Theorem} \end{cases}$$

— OR —

$$2 : \begin{cases} 1 : v_P(0.3) > v_P(1.7) \\ \quad \text{and } v_P(1.7) < v_P(2.8) \\ 1 : \text{justification, using} \\ \quad \text{Extreme Value Theorem} \end{cases}$$

(b)
$$\begin{aligned} \int_0^{2.8} v_P(t)\,dt &\approx 0.3\left(\frac{v_P(0) + v_P(0.3)}{2}\right) + 1.4\left(\frac{v_P(0.3) + v_P(1.7)}{2}\right) \\ &\quad + 1.1\left(\frac{v_P(1.7) + v_P(2.8)}{2}\right) \\ &= 0.3\left(\frac{0 + 55}{2}\right) + 1.4\left(\frac{55 + (-29)}{2}\right) + 1.1\left(\frac{-29 + 55}{2}\right) \\ &= 40.75 \end{aligned}$$

1 : answer, using trapezoidal sum

(c) $v_Q(t) = 60 \Rightarrow t = A = 1.866181$ or $t = B = 3.519174$
$v_Q(t) \ge 60$ for $A \le t \le B$

$$\int_A^B \left|v_Q(t)\right|\,dt = 106.108754$$

The distance traveled by particle Q during the interval $A \le t \le B$ is 106.109 (or 106.108) meters.

$$3 : \begin{cases} 1 : \text{interval} \\ 1 : \text{definite integral} \\ 1 : \text{distance} \end{cases}$$

(d) From part (b), the position of particle P at time $t = 2.8$ is
$x_P(2.8) = \int_0^{2.8} v_P(t)\,dt \approx 40.75.$

$$x_Q(2.8) = x_Q(0) + \int_0^{2.8} v_Q(t)\,dt = -90 + 135.937653 = 45.937653$$

Therefore at time $t = 2.8$, particles P and Q are approximately $45.937653 - 40.75 = 5.188$ (or 5.187) meters apart.

$$3 : \begin{cases} 1 : \int_0^{2.8} v_Q(t)\,dt \\ 1 : \text{position of particle } Q \\ 1 : \text{answer} \end{cases}$$

Question 3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$\Rightarrow\ 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$

$\Rightarrow\ \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$

$3: \begin{cases} 1: \int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \\ 1: \int_{-2}^{5} f(x)\,dx \\ 1: \text{answer} \end{cases}$

(b) $\int_3^5 (2f'(x) + 4)\,dx = 2\int_3^5 f'(x)\,dx + \int_3^5 4\,dx$
$= 2(f(5) - f(3)) + 4(5-3)$
$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$
$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$

— OR —

$\int_3^5 (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$
$= (2f(5) + 20) - (2f(3) + 12)$
$= (2\cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$
$= 2 + 2\sqrt{5}$

$2: \begin{cases} 1: \text{Fundamental Theorem of Calculus} \\ 1: \text{answer} \end{cases}$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$3: \begin{cases} 1: g'(x) = f(x) \\ 1: \text{identifies } x = -1 \text{ as a candidate} \\ 1: \text{answer with justification} \end{cases}$

(d) $\lim_{x\to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$
$= \frac{10 - 3\cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$

1 : answer

Question 4

(a) $V = \pi r^2 h = \pi (1)^2 h = \pi h$

$$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

$2 : \begin{cases} 1 : \dfrac{dV}{dt} = \pi \dfrac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$

(b) $\dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} = -\dfrac{1}{20\sqrt{h}} \cdot \left(-\dfrac{1}{10}\sqrt{h}\right) = \dfrac{1}{200}$

Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$3 : \begin{cases} 1 : \dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}} \\ 1 : \dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$

(c) $\dfrac{dh}{\sqrt{h}} = -\dfrac{1}{10}\,dt$

$\displaystyle\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$

$2\sqrt{h} = -\dfrac{1}{10}t + C$

$2\sqrt{5} = -\dfrac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$

$2\sqrt{h} = -\dfrac{1}{10}t + 2\sqrt{5}$

$h(t) = \left(-\dfrac{1}{20}t + \sqrt{5}\right)^2$

$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \quad\ \text{and uses initial condition} \\ 1 : h(t) \end{cases}$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

Question 5

(a) $\int_0^2 (h(x) - g(x))\,dx = \int_0^2 \left(\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\frac{\pi}{2}x\right)\right)\right) dx$

$$= \left[\left(6x - \frac{2}{3}(x-1)^3\right) - \left(-2x + \frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2}$$

$$= \left(\left(12 - \frac{2}{3}\right) - (-4 + 0)\right) - \left(\left(0 + \frac{2}{3}\right) - (0 + 0)\right)$$

$$= 12 - \frac{2}{3} + 4 - \frac{2}{3} = \frac{44}{3}$$

The area of R is $\frac{44}{3}$.

$4 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative of } 3\cos\left(\frac{\pi}{2}x\right) \\ 1 : \text{antiderivative of remaining terms} \\ 1 : \text{answer} \end{cases}$

(b) $\int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx$

$$= \left[\ln(x+3)\right]_{x=0}^{x=2} = \ln 5 - \ln 3$$

The volume of the solid is $\ln 5 - \ln 3$.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $\pi\int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2\right) dx$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{form of integrand} \\ 1 : \text{integrand} \end{cases}$

Question 6

(a) $h'(2) = \frac{2}{3}$

1 : answer

(b) $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$

$a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$

$3 : \begin{cases} 1 : \text{form of product rule} \\ 1 : a'(x) \\ 1 : a'(2) \end{cases}$

(c) Because h is differentiable, h is continuous, so $\lim_{x \to 2} h(x) = h(2) = 4$.

Also, $\lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3}$, so $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$.

Because $\lim_{x \to 2} \left(x^2 - 4\right) = 0$, we must also have $\lim_{x \to 2} \left(1 - (f(x))^3\right) = 0$.

Thus $\lim_{x \to 2} f(x) = 1$.

Because f is differentiable, f is continuous, so $f(2) = \lim_{x \to 2} f(x) = 1$.

Also, because f is twice differentiable, f' is continuous, so $\lim_{x \to 2} f'(x) = f'(2)$ exists.

Using L'Hospital's Rule,

$\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = \lim_{x \to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$

Thus $f'(2) = -\frac{1}{3}$.

$4 : \begin{cases} 1 : \lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4 \\ 1 : f(2) \\ 1 : \text{L'Hospital's Rule} \\ 1 : f'(2) \end{cases}$

(d) Because g and h are differentiable, g and h are continuous, so $\lim_{x \to 2} g(x) = g(2) = 4$ and $\lim_{x \to 2} h(x) = h(2) = 4$.

Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim_{x \to 2} k(x) = 4$.

Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$.

Thus k is continuous at $x = 2$.

1 : continuous with justification